Journal of Algebra **238**, 762–775 (2001)
doi:10.1006/jabr.2000.8636, available online at http://www.idealibrary.com on IDEAL®

Weights of Markov Traces on Cyclotomic Hecke Algebras

Hebing Rui[1]

metadata, citation and similar papers at core.ac.uk

Communicated by Gordon James

Received May 10, 2000

In [J], Jones used the Markov traces on the Hecke algebras of type A to construct the knot invariants. Motivated by Jones's work, Lambropoulou [L] introduced the Markov traces on the cyclotomic Hecke algebras of type $G(m, 1, r)$ (see [GL] for the case $m = 2$). Since any linear trace function can be expressed as a linear combination of the irreducible characters, where the coefficients are called weights, it is natural to ask how to determine the weights of the Markov traces.

In [W1], Wenzl proved that the weights of the Markov traces on the Hecke algebras of type A (i.e., $m = 1$) can be expressed via Schur functions (see (3.3)). In [O], Orellana proved that there is an epimorphism from the Hecke algebra of type B (i.e., $m = 2$) with special parameters to some reduced algebra of a type-A Hecke algebra. This enabled her to use Wenzl's result to determine the weights of the Markov traces on type-B Hecke algebras. In [I], Iancu gave a conjecture about the general weight formulas of the type-B Hecke algebras.

The main purpose of this paper is to determine the weights of the Markov traces on the cyclotomic Hecke algebras of type $G(m, 1, r)$, generalizing the results in [O]. Our arguments are based on those in [loc. cit]. However, we will not use the results on *Jones basic construction*.

The content of this paper is organized as follows. In Section 1, we collect some of the results on type-A Hecke algebras. We discuss the cyclotomic Hecke algebras in Section 2. The main result of this section is the

[1] Research was supported by the National Natural Science Foundation 19871055, Foundation for University Key Teacher by the Ministry of Education, and the 973 Project of China. The paper was written while the author was visiting the International Center for Theoretical Physics, Trieste, Italy. He thanks ICTP for its hospitality during his visit.

0021-8693/01 $35.00
Copyright © 2001 by Academic Press
All rights of reproduction in any form reserved.

existence of the epimorphism from a cyclotomic Hecke algebra with special parameters to some reduced Hecke algebra of type A. We determine the weights of the Markov traces on the cyclotomic Hecke algebras in Section 3.

After the paper was completed, the author was kindly informed that Geck *et al.* [GIM] independently obtained results similar to (3.12). Moreover, they extended them to general choices of parameters.

1. HECKE ALGEBRA OF TYPE A_{r-1}

In this section, we collect some of the results on the Hecke algebras of type A.

Let $\mathfrak{S}_r$ be the symmetry group on r letters. Let $\mathbb{C}$ be the complex field with nonzero element $q \in \mathbb{C}$. The Hecke algebra $\mathscr{H}_r$ associated to $\mathfrak{S}_r$ is an associative algebra over $\mathbb{C}$ with generators T_i subject to the conditions

$$\begin{aligned}
&(1) \qquad (T_i - q)(T_i + 1) = 0, && 1 \le i \le r-1,\\
&(2) \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, && 1 \le i \le r-2,\\
&(3) \qquad T_i T_j = T_j T_i, && 1 \le i < j-1 \le r-2.
\end{aligned} \tag{1.1}$$

Let $e(q) \in \mathbb{Z}$ be the minimal integer n such that

$$1 + q + \cdots + q^{n-1} = 0. \tag{1.2}$$

If such an integer n does not exist, then set $e(q) = +\infty$. It is known that $\mathscr{H}_r$ is semisimple if and only if $e(q) > r$.

From here onward, we always assume that $e(q) = +\infty$ since we are interested in the semisimple Hecke algebras.

The nonisomorphic simple $\mathscr{H}_r$-modules V_λ are indexed by the set $\Lambda^+(r)$ of partitions of r (say a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition of r if $|\lambda| = \sum_i \lambda_i = r$). There are several ways to construct the simple $\mathscr{H}_r$-modules (see, e.g., [KL, DJ, and M]). In this paper, we recall the construction of V_λ due to Wenzl in [W1].

For each $\lambda \in \Lambda^+(r)$, one can identify λ with the Young diagram $\mathscr{Y}(\lambda)$, which consists of boxes arranged in a manner as illustrated by the example $\lambda = (3, 2)$ for which we have

$$\mathscr{Y}(\lambda) = \boxed{\begin{matrix} \square & \square & \square \\ \square & \square & \end{matrix}}\,.$$

A λ tableau $\mathbf{t}$ is obtained by replacing each box with one of the numbers $1, 2, \dots, r$, allowing no repeats. The λ-tableau $\mathbf{t}$ is called standard if the entries are increasing along each row and each column. Let $T^s(\lambda)$ be the set of all standard λ-tableaux.

For each number i, $1 \le i \le r$, let $c(\mathbf{t}, i) = j - k$ if i is in the jth column and kth row of $\mathbf{t} \in T^s(\lambda)$. Let $d(i+1, i) = c(\mathbf{t}, i+1) - c(\mathbf{t}, i)$. Then V_λ is a vector space over $\mathbb{C}$ with basis $v_{\mathbf{t}}$, $\mathbf{t} \in T^s(\lambda)$, and the action of T_i, $1 \le i \le r-1$, is given as

$$T_i v_{\mathbf{t}} = a_d(q) v_{\mathbf{t}} + c_d(q) v_{s_i \mathbf{t}}, \tag{1.3}$$

where $d = d(i+1, i)$, $a_d(q) = q^d(1-q)/1 - q^d$, $c_d(q) = \{(1-q^{d+1})(1-q^{d-1})\}^{1/2}(1-q^d)^{-1}$, and $s_i\mathbf{t}$ is the tableau obtained from $\mathbf{t}$ by switching i and $i+1$. It is easy to see that $s_i\mathbf{t}$ is not standard only if i, $i+1$ are either in the same column or in the same row of $\mathbf{t}$. However, $d = \pm 1$ and $c_d = 0$ in this case.

Let $T_{r,1} = T_{r-1} \cdots T_1$. The *full-twist element* $\Delta_r^2 := (T_{r,1})^r$ is in the center of $\mathscr{H}_r$. In [W2, 3.2.1], Wenzl proved that Δ_r^2 acts on V_λ as the scalar c_λ and

$$c_\lambda = q^{r(r-1) - \sum_{i<j}(\lambda_i + 1)\lambda_j} \qquad \text{for } \lambda = (\lambda_1, \lambda_2, \dots) \in \Lambda^+(r). \tag{1.4}$$

For each standard λ-tableau $\mathbf{t} \in T^s(\lambda)$, Wenzl [W1, (2.7)] introduced a minimal idempotent $p_{\mathbf{t}} \in \mathscr{H}_r$, called the path idempotent with respect to $\mathbf{t}$. Let z_λ be the *minimal central idempotent* of $\mathscr{H}_r$ with respect to V_λ. Let $\mathbf{t}'$ be the standard tableau obtained from $\mathbf{t}$ by removing the box containing r. Then

$$p_{[1]} = 1 \qquad \text{and} \qquad p_{\mathbf{t}} = z_\lambda p_{\mathbf{t}'}. \tag{1.5}$$

It is proved in [W1, Corollary 2.3] that

$$p_{\mathbf{s}} p_{\mathbf{t}} = \delta_{\mathbf{st}} p_{\mathbf{s}} \qquad \text{for any } \mathbf{s}, \mathbf{t} \in T^s(\lambda), \lambda \in \Lambda^+(r). \tag{1.6}$$

For $\mathbf{t} \in T^s(\mu)$, $\mu \in \Lambda^+(r)$, the subalgebra $p_{\mathbf{t}} \mathscr{H}_{r+f} p_{\mathbf{t}}$ of $\mathscr{H}_{f+r}$ is called the *reduced algebra* of $\mathscr{H}_{r+f}$ with respect to $\mathbf{t}$ in [O]. It is known that $p_{\mathbf{t}} \mathscr{H}_{r+f} p_{\mathbf{t}}$ is semisimple if $\mathscr{H}_{f+r}$ is semisimple (see, e.g., [Ma, Chap. 1, Example 12]). Let V_λ, $\lambda \in \Lambda^+(f+r)$, be the simple $\mathscr{H}_{f+r}$-module defined as above. Using the ordinary branching rule (see, e.g., [W1, (2.6)]) for V_λ and (1.5), one will see immediately that

$$p_{\mathbf{t}} V_\lambda = 0 \text{ unless } \mu \subseteq \lambda. \tag{1.7}$$

Recall that, for two partitions λ, μ, we write $\mu \subseteq \lambda$ and say μ is contained in λ if $\mu_i \le \lambda_i$ for every i. Suppose $\mu \subseteq \lambda$. Say $\mathbf{s} \in T^s(\lambda)$ contains $\mathbf{t} \in T^s(\mu)$ if $\mathbf{t}$ is obtained from $\mathbf{s}$ by dropping the boxes containing $|\mu| + 1, \dots, |\lambda|$. By (1.3) and (1.5),

$$p_{\mathbf{t}} v_{\mathbf{s}} = \begin{cases} v_{\mathbf{s}}, & \mathbf{t} \subseteq \mathbf{s}, \\ 0, & \text{otherwise}. \end{cases} \tag{1.8}$$

Thus,

$$p_{\mathbf{t}}V_\lambda \text{ is spanned by } v_{\mathbf{s}}, \text{ where } \mathbf{s} \text{ are standard } \lambda\text{-tableaux containing } \mathbf{t}. \tag{1.9}$$

The complete set of nonisomorphic simple $p_{\mathbf{t}}\mathscr{H}_{f+r}p_{\mathbf{t}}$-modules is $p_{\mathbf{t}}V_\lambda$, $\lambda \in \Lambda^+(f+r)$, with $\mu \subseteq \lambda$. This follows directly from [Ma, Chap. 1, Example 12].

2. CYCLOTOMIC HECKE ALGEBRA OF TYPE $G(m, 1, r)$

Let $W = (\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_r$ be the wreath product of the cyclic group of order m and the symmetric group $\mathfrak{S}_r$. Then W is not a Coxeter group except for $m = 1, 2$. Let $\mathbb{C}$ be the complex field with nonzero elements $q, u_1, \dots, u_m$. The cyclotomic Hecke algebra $\mathbf{H}_r$ associated to W [AK, BM] is an associative algebra over $\mathbb{C}$ with generators $T_0, T_1, \dots, T_{r-1}$ subject to the conditions

$$\begin{array}{ll} (1) & (T_0 - u_1)\cdots(T_0 - u_m) = 0 \\ (2) & T_0T_1T_0T_1 = T_1T_0T_1T_0, \\ (3) & (T_i - q)(T_i + 1) = 0, \qquad 1 \le i \le r-1, \\ (4) & T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad 1 \le i \le r-2, \\ (5) & T_iT_j = T_jT_i, \qquad 1 \le i < j-1 \le r-2. \end{array} \tag{2.1}$$

It is known that $\mathbf{H}_r$ is the Hecke algebra of type A_{r-1} (see Section 1), resp. B_r, if $m = 1$, resp. $m = 2$, and $u_1 = Q$, $u_2 = -1$. Also, when $q = 1$ and $u_i = \xi^i$ where ξ is the primitive mth root of unity, $\mathbf{H}_r$ is the group algebra of the complex reflection group of type $G(m, 1, r)$ over $\mathbb{C}$.

Remark. In [AK] and [BM], $\mathbf{H}_r$ was defined over arbitrary commutative ring R without the assumption $u_i^{-1} \in R$. Since the main purpose of this paper is to study the Markov traces on cyclotomic Hecke algebras over the complex field $\mathbb{C}$, we have to add the assumption $u_i^{-1} \in \mathbb{C}$ (see [L, Section 4]).

We can assume $u_m = -1$ without loss of any generality since we get an algebra with parameters $q, -u_1u_m^{-1}, \dots, -u_{m-1}u_m^{-1}, -1$, which is isomorphic to the original one if we replace T_0 with $-u_mT_0$ in (2.1). Let

$$f_{m,r} = \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} \prod_{k=1-r}^{r-1} \left(u_iq^k - u_j\right). \tag{2.2}$$

Let $e(q) \in \mathbb{Z}$ be defined as in (1.2). The following result is the special case of the main theorem of [A1]. See also [DR1, (5.2)]

(2.3) *Criterion of Semisimplicity.* Let $\mathbf{H}_r$ be the cyclotomic Hecke algebra of type $G(m,1,r)$ over the field $\mathbb{C}$ with parameters q, q^{-1}, u_i, $i = 1, \dots, m$. Then $\mathbf{H}_r$ is semisimple if and only if $f_{m,r} \neq 0$ and $e(q) > r$.

From here onwards, we always assume $f_{m,r} \neq 0$ and $e(q) = +\infty$, which says that $\mathbf{H}_r$ is semisimple.

Fix positive integers l, n_i with $l > r$ and $n_i > r$, $1 \le i \le m-1$. Let $\tilde{\mathbf{H}}_r$ be the cyclotomic Hecke algebra with nonzero parameters $q, \tilde{u}_i, 1 \le i \le m$, such that $e(q) = +\infty$ and

$$\tilde{u}_i = -q^{(m-i)l + \sum_{j=i}^{m-1} n_j}, \; i = 1, \dots, m-1, \qquad \text{and} \qquad \tilde{u}_m = -1. \quad (2.4)$$

If $j > i$, then $\tilde{u}_i q^k - \tilde{u}_j \neq 0$ since $(j-i)l + n_i + \cdots + n_{j-1} + k > 0$ for any $1 - r \le k \le r - 1$ and $e(q) = +\infty$. Thus $f_{m,r} \neq 0$, and consequently $\tilde{H}_r$ is semisimple.

The main result of this section is that there is a surjective algebraic homomorphism from $\tilde{\mathbf{H}}_r$ to a certain reduced algebra of the Hecke algebra of type A. A by product is a construction of simple $\tilde{\mathbf{H}}_r$-modules. The result for $m = 2$ is due to Orellana [O, Section 3]. We point out that the simple $\mathbf{H}_r$-modules over an arbitrary field have been classified in [AK, DJM] for the semisimple case, in [DR2] under the assumption $f_{m,r} \neq 0$, and in [A2] in general.

We need some notation. Denote

$$\alpha = \Big(\underbrace{(m-1)l, \dots, (m-1)l}_{n_1}, \dots, \underbrace{l, \dots, l}_{n_{m-1}}\Big). \quad (2.5)$$

Then $\alpha \in \Lambda^+(f)$, $f = \sum_{i=1}^{m-1} n_i(m-i)l$. Let $\lambda \in \Lambda^+(f+r)$ with $\alpha \subseteq \lambda$. The set difference $\lambda - \alpha$ is known as the skew diagram λ/α. One can identify $\lambda - \alpha$ with a sequence of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)})$, called a multipartition or an m-partition of r. Since $n_i > r$, $1 \le i \le m-1$, the length of the partition $\lambda^{(i)}$ is strictly less than n_i (say n is the length of λ if n is the maximal index i with $\lambda_i \neq 0$). So there is a bijection between the set $\Lambda_m^+(r)$ of all m-partitions of r and the set $\tilde{\Lambda}^+(f+r)$ of all partitions of $f+r$, which contain α defined as in (2.5). More explicitly, for $\lambda \in \tilde{\Lambda}_m^+(f+r)$ there is a unique $\boldsymbol{\lambda} \in \Lambda_m^+(r)$ determined uniquely by the skew diagram

$\lambda - \alpha$, such that

$$\lambda = \tilde{\boldsymbol{\lambda}} = \Big(\underbrace{(m-1)l + \lambda_1^{(1)}, \dots, (m-1)l + \lambda_{n_1}^{(1)}}_{n_1}, \dots, \underbrace{\lambda_1^{(m)}, \dots, \lambda_{n_m}^{(m)}}_{n_m}\Big). \tag{2.6}$$

Here n_m is part of the $\lambda^{(m)}$. Recall that λ can be identified with μ if μ is obtained from λ by adding or deleting some zeroes at the end of λ. For example, $(2, 0)$ can be identified with (2) and $(2, 0, 0)$, etc. So, (2.6) is valid for all $\boldsymbol{\lambda} \in \Lambda_m^+(r)$.

Let $\mathbf{t}^\lambda$ be the standard λ-tableau in which the numbers $1, \dots, |\lambda|$ appear in order along successive columns. For example,

$$\mathbf{t}^\lambda = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix}, \qquad \text{if } \lambda = (32).$$

Let $\lambda \in \tilde{\Lambda}^+(1+f)$. Since $\alpha \subseteq \lambda$, $\lambda = \gamma^{(i)}$, $1 \le i \le m$, where

$$\gamma^{(i)} = \tilde{1}_i \qquad \text{and} \qquad 1_i = ((0), \dots, (0), (1_i), (0), \dots, (0)) \in \Lambda_m^+(1). \tag{2.7}$$

(2.8) THEOREM. *Keep the notations above. Let $\mathbf{t} = \mathbf{t}^\alpha$, where α is defined as in* (2.5).

(1) *There is an epimorphism $\rho_{f,r}\colon \tilde{\mathbf{H}}_r \to p_{\mathbf{t}} \mathscr{H}_{r+f} p_{\mathbf{t}}$ subject to the conditions*

$$\rho_{f,r}(1) = p_{\mathbf{t}},$$

$$\rho_{f,r}(T_0) = -\frac{c_\alpha}{c_{\gamma(m)}} p_{\mathbf{t}} \Delta_f^{-2} \Delta_{f+1}^2,$$

$$\rho_{f,r}(T_i) = p_{\mathbf{t}} T_{i+f}, \qquad i = 1, \dots, r-1.$$

(2) *For each $\boldsymbol{\lambda} \in \Lambda_m^+(r)$, $p_{\mathbf{t}} V_{\tilde{\boldsymbol{\lambda}}}$ is a simple $\tilde{\mathbf{H}}_r$-module and the set $\{p_{\mathbf{t}} V_{\tilde{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \Lambda_m^+(r)\}$ forms the complete set of nonisomorphic $\tilde{\mathbf{H}}_r$-modules.*

Proof. The proof of (2) will be included in the proof of (1). To show that $\rho_{f,r}$ is a homomorphism of an algebra, we only need to verify the map $\rho_{f,r}$ preserving the defining relations (2.1(1–5)). Since $p_{\mathbf{t}} \in \mathscr{H}_f$, it commutes with T_{i+f} for $i \ge 1$, and the relations (2.1(3–5)) follows. On the other hand, the full-twist element Δ_f^2 is in the center of $\mathscr{H}_f$ and $\Delta_f^{-2}\Delta_{f+1}^2 = T_{f+1,1} T_{1,f+1}$. So 2.1(2) follows easily from the braid relations given in (1.1(2–3)). We claim that $\rho_{f,r}(T_0) \in p_{\mathbf{t}} \mathscr{H}_{f+1} p_{\mathbf{t}}$ acts on the simple

$p_{\mathbf{t}}\mathscr{H}_{f+1}p_{\mathbf{t}}$-modules $p_{\mathbf{t}}V_{\gamma^{(i)}} \subseteq V_{\gamma^{(i)}}$, $1 \le i \le m$, as scalar $\tilde{u}_i$, $1 \le i \le m$. Because $p_{\mathbf{t}}$ is the identity element in the reduced algebra $p_{\mathbf{t}}\mathscr{H}_{r+f}p_{\mathbf{t}}$, the claim implies $\prod_{i=1}^{m}(\rho_{f,r}(T_0) - \tilde{u}_i p_{\mathbf{t}}) = 0$, proving 2.1(1).

By (1.3)–(1.4) and the ordinary branching rule for $V_{\gamma^{(i)}}$, Δ_f^2 acts on $p_{\mathbf{t}}V_{\gamma^{(i)}}$ as scalar c_α. So, $\rho_{f,r}(T_0)$ acts on $p_{\mathbf{t}}V_{\gamma^{(i)}}$ as the scalar $-c_{\gamma^{(i)}}/c_{\gamma^{(m)}}$. A direct computation shows that

$$-\frac{c_{\gamma^{(i)}}}{c_{\gamma^{(m)}}} = \begin{cases} \tilde{u}_i = -q^{(m-i)l+\sum_{j=i}^{m-1} n_j}, & \text{if } i \ne m, \\ \tilde{u}_m = -1, & \text{if } i = m, \end{cases} \tag{2.9}$$

proving our claim. Obviously, each $p_{\mathbf{t}}\mathscr{H}_{f+r}p_{\mathbf{t}}$-module is an $\tilde{\mathbf{H}}_r$-module. In particular, $p_{\mathbf{t}}V_{\tilde{\boldsymbol{\lambda}}}$ is an $\tilde{\mathbf{H}}_r$-module, too. If (2) holds, then $\rho_{f,r}$ is surjective by comparing the dimensions of im $\rho_{f,r}$ and $p_{\mathbf{t}}\mathscr{H}_{f+r}p_{\mathbf{t}}$ via the Wedderburn Theorem for a finite-dimensional semisimple algebra over a field.

Now, we prove (2) by induction on r. Assume $r = 1$. Since $e(q) = +\infty$, $\tilde{u}_i \ne \tilde{u}_j$, for any $1 \le i < j \le m$ (see (2.4)), and $p_{\mathbf{t}}V_{\gamma^{(i)}} \not\cong p_{\mathbf{t}}V_{\gamma^{(j)}}$, $i \ne j$. Since there is only one standard $\gamma^{(i)}$-tableau containing $\mathbf{t}^\alpha$, $\dim p_{\mathbf{t}}V_{\gamma^{(i)}} = 1$. By [AK, (3.10)], $\dim \tilde{\mathbf{H}}_1 = m = \sum_{i=1}^{m}(\dim p_{\mathbf{t}}V_{\gamma^{(i)}})^2$. By the Wedderburn Theorem, $\{p_{\mathbf{t}}V_{\gamma^{(i)}} \mid i = 1, \dots, m\}$ forms the complete set of nonisomorphic $\tilde{\mathbf{H}}_1$-modules.

In order to deal with the case $r > 1$, we need the notion of a standard $\boldsymbol{\lambda}$-tableau for $\boldsymbol{\lambda} \in \Lambda_m^+(r)$. Say a sequence of tableaux $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_m)$ is a $\boldsymbol{\lambda}$-tableau for $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in \Lambda_m^+(r)$ if it is obtained from the sequence of the Young diagrams $\mathscr{Y}(\boldsymbol{\lambda}) = (\mathscr{Y}(\lambda^{(1)}), \dots, \mathscr{Y}(\lambda^{(m)}))$ by replacing each box with one of the numbers $1, 2, \dots, \sum_{i=1}^{m}|\lambda^{(i)}|$, allowing no repeats. Such a tableau $\mathbf{u}$ is called standard if the numbers appear to be increasing along each column and each row of each subtableau $\mathbf{u}_i$, $1 \le i \le m$.

Let $\mathbf{s} \in T^s(\tilde{\boldsymbol{\lambda}})$ with $\mathbf{t}^\alpha = \mathbf{t} \subseteq \mathbf{s}$. If we replace the entry a in the skew tableau $\mathbf{s} - \mathbf{t}$ by $a - f$, one will get a standard $\boldsymbol{\lambda}$-tableau and vice versa. In particular, $\#T^s(\tilde{\boldsymbol{\lambda}}/\alpha) = \#T^s(\boldsymbol{\lambda})$, where $T^s(\tilde{\boldsymbol{\lambda}}/\alpha) = \{\mathbf{s} - \mathbf{t} \mid \mathbf{s} \in T^s(\tilde{\boldsymbol{\lambda}})\}$. By [AK, (3.10)] or [DJM, (3.30)],

$$\dim \tilde{\mathbf{H}} = \sum_{\lambda \in \Lambda_m^+(r)} \#T^s(\boldsymbol{\lambda})^2 = \sum_{\lambda \in \Lambda_m^+(r)} (\dim p_{\mathbf{t}}V_{\tilde{\boldsymbol{\lambda}}})^2. \tag{2.10}$$

Now, the case $r > 1$ follows from (2.10) and the Wedderburn Theorem together with the results (a) and (b) which follow.

(a) $p_{\mathbf{t}}V_{\tilde{\boldsymbol{\lambda}}} \not\cong p_{\mathbf{t}}V_{\tilde{\boldsymbol{\lambda}}_1}$, for any $\boldsymbol{\lambda}, \boldsymbol{\lambda}_1 \in \Lambda_m^+(r)$ with $\boldsymbol{\lambda} \ne \boldsymbol{\lambda}_1$.

(b) For each $\boldsymbol{\lambda} \in \Lambda^+(r)$, $p_{\mathbf{t}}V_{\tilde{\boldsymbol{\lambda}}}$ is simple.

Using the ordinary branching rule for $V_{\tilde{\boldsymbol\lambda}}$, we have

$$p_{\mathbf{t}}V_{\tilde{\boldsymbol\lambda}} \cong \bigoplus_{\mu} p_{\mathbf{t}}V_{\tilde{\boldsymbol\mu}} \tag{2.11}$$

where $\boldsymbol\mu$ ranges over all m-partitions of $r+f-1$, which are obtained from $\boldsymbol\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ by deleting one box $\mathfrak{n}$ from some $\mathscr{Y}(\lambda^{(i)})$, $1 \le i \le m$. In this case, we write $\boldsymbol\mu \overset{\mathfrak{n}}{\to} \boldsymbol\lambda$.

If $\boldsymbol\lambda \neq \boldsymbol\lambda_1$, $\boldsymbol\lambda$, $\boldsymbol\lambda_1 \in \Lambda_m^+(r)$, then $p_{\mathbf{t}}V_{\tilde{\boldsymbol\lambda}} \not\cong p_{\mathbf{t}}V_{\tilde{\boldsymbol\lambda}_1}$ since they have different decompositions of $\tilde{\mathbf{H}}_{r-1}$-modules.

Let $\boldsymbol\mu$ be an m-partition of $r-1$ with $\boldsymbol\mu \overset{\mathfrak{n}}{\to} \boldsymbol\lambda$. If such a $\boldsymbol\mu$ is unique, then (2.11) forces $p_{\mathbf{t}}V_{\tilde{\boldsymbol\lambda}}$ to be a simple $\tilde{\mathbf{H}}_{r-1}$-module. It must be a simple $\tilde{\mathbf{H}}_r$-module, too. Suppose that the m-partition $\boldsymbol\mu$ is not unique. Let W be a $\tilde{\mathbf{H}}_r$-submodule of $p_{\mathbf{t}}V_{\tilde{\boldsymbol\lambda}}$. Then W contains a simple $\tilde{\mathscr{H}}_{f+r-1}$-module $V_{\boldsymbol\mu}$. Let $\boldsymbol\mu_1 \in \Lambda_m^+(r-1)$ with $\boldsymbol\mu \neq \boldsymbol\mu_1$ and $\boldsymbol\mu_1 \overset{\mathfrak{m}}{\to} \boldsymbol\lambda$. Take a standard $\tilde{\boldsymbol\lambda}$-tableau $\mathbf{t}$ such that $r+f-1$, resp. $r+f$, is in the box $\mathfrak{m}$, resp. $\mathfrak{n}$. Obviously, $r+f$ and $r+f-1$ are neither in the same row nor in the same column. So, $d = d(i+1, i) \neq \pm 1$ and $c_d \neq 0$. By (1.3),

$$\rho_{r,f}(T_{r-1})v_{\mathbf{t}} = a_d(q)v_{\mathbf{t}} + c_d(q)v_{s_{r+f-1}\mathbf{t}} \in W.$$

So, $v_{s_{r+f-1}\mathbf{t}} \in W \cap p_{\mathbf{t}}V_{\boldsymbol\mu_1} \subset W$, forcing $W = p_{\mathbf{t}}V_{\tilde{\boldsymbol\lambda}}$. ∎

3. WEIGHTS OF THE MARKOV TRACES

In this section we determine the weights of the Markov traces on the cyclotomic Hecke algebras $\mathbf{H}_r$. Let $\mathbf{H}_r$ (resp. $\tilde{\mathbf{H}}_r$) be the semisimple cyclotomic Hecke algebra of type $G(m,1,r)$ over the complex field $\mathbb{C}$ (resp., with special parameters $q, \tilde{u}_i$, $i = 1, \dots, m$ given in (2.4)). Then the algebra $\mathbf{H}_{r-1}$ (resp. $\tilde{\mathbf{H}}_{r-1}$) can be embedded naturally into $\mathbf{H}_r$ (resp. $\tilde{\mathbf{H}}_r$). Let $\mathbf{H} = \bigcup_{r=1}^{+\infty} \mathbf{H}_r$ and $\tilde{\mathbf{H}} = \bigcup_{r=1}^{+\infty} \tilde{\mathbf{H}}_r$.

(3.1) THEOREM [L, Theorem 6]. *Given* z, $s_k \in \mathbb{C}(q, u_1^{\pm 1}, \dots, u_m^{\pm 1})$ *with* $0 \le k \le m-1$. *Then there is a linear function* tr: $\mathbf{H} \to \mathbb{C}(q^{\pm 1}, u_1^{\pm 1}, \dots, u_m^{\pm 1}, z, s_k)$, $0 \le k \le m$, *determined uniquely by the rules*

(1) $\operatorname{tr}(ab) = \operatorname{tr}(ba)$, *for* $a, b \in \mathbf{H}_r$,

(2) $\operatorname{tr}(1) = 1$, *for all* $\mathbf{H}_r$,

(3) $\operatorname{tr}(aT_r) = z\operatorname{tr}(a)$, *for* $a \in \mathbf{H}_r$,

(4) $\operatorname{tr}\left(at_r'^{k}\right) = s_k \operatorname{tr}(a)$, *for* $a \in \mathbf{H}_r, 0 \le k \le m-1$,

where $t_r'^{k} = T_{r+1,1}T_0^kT_{r+1,1}^{-1}$.

As mentioned in Section 2, we can assume $u_m = -1$. If one hopes to get the formula involving u_m with $u_m \neq -1$, one should use $-u_m^{-1}u_i$ to replace u_i.

The trace function defined above is known as the Markov trace on $\mathbf{H}$ with respect to the parameters z, s_k, $0 \leq k \leq m-1$. Note that any linear trace function on a finite-dimensional algebra is a linear combination of the irreducible characters, where the coefficients are called the weights. By [AK, (3.10)] or [DJM, (3.30)],

$$\operatorname{tr}|_{\mathbf{H}_r}(x) = \sum_{\boldsymbol{\lambda} \in \Lambda_m^+(r)} \omega_{\boldsymbol{\lambda}} \chi^{\boldsymbol{\lambda}}(x), \qquad \text{for } x \in \mathbf{H}_r, \tag{3.2}$$

where $\chi^{\boldsymbol{\lambda}}$ is the character of the irreducible representation $\pi_{\boldsymbol{\lambda}}$ of $\mathbf{H}_r$ indexed by the m-partition $\boldsymbol{\lambda} \in \Lambda_m^+(r)$. In fact, since the construction of simple $\mathbf{H}_r$-modules is independent of the value $q, u_1, \dots, u_m$ under the assumption (2.3) (see, e.g., [DJM, (3.30)] or [DR2, (2.1)]), we can assume that $\pi_{\boldsymbol{\lambda}}$ corresponds to $p_{\mathbf{t}} V_{\tilde{\boldsymbol{\lambda}}}$ when $u_i = \tilde{u}_i$, $1 \leq i \leq m$.

Now, we recall some of the results on the Markov trace, which is defined on the Hecke algebras of type A. In [W1, Theorem 3.6], Wenzl computed the weight w_λ for $\lambda \in \Lambda^+(r)$ when $\mathbf{H}_r$ is the Hecke algebra of type A_{r-1} and $z = q^n(1-q)/(1-q^n)$, $n \in \mathbb{N}$. More explicitly,

$$\begin{aligned} \omega_\lambda &= s_{\lambda, n} = \frac{s_\lambda(1, q, \dots, q^{n-1})}{s_{[1]}(1, q, \dots, q^{n-1})^{|\lambda|}}, \\ s_\lambda(1, q, \dots, q^{n-1}) &= q^{\sum_{i=1}^{l(\lambda)}(i-1)\lambda_i} \cdot \prod_{1 \leq i < j \leq n} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{1 - q^{j-i}}, \end{aligned} \tag{3.3}$$

where s_λ is the Schur function [Mc] and $l(\lambda)$ is the length of λ (i.e., the maximal index with $\lambda_{l(\lambda)} \neq 0$). It is proved in [W1, Lemma 3.5] that

$$s_{\lambda, n} = 0, \qquad \text{for } l(\lambda) > n.$$

By (1.8), $p_{\mathbf{t}} v_{\mathbf{s}} = \delta_{\mathbf{t}, \mathbf{s}} v_{\mathbf{s}}$ for any $\mathbf{s} \in T^s(\alpha)$. Thus $\chi^\lambda(p_{\mathbf{t}}) = \delta_{\lambda, \alpha}$ and

$$\operatorname{tr}(p_{\mathbf{t}}) = \operatorname{tr}|_{\mathscr{H}_f}(p_{\mathbf{t}}) = w_\alpha = s_{\alpha, n}. \tag{3.4}$$

In order to get the nonzero element $s_{\alpha, n}$, we have to assume $n \geq \sum_{i=1}^{m-1} n_i$ since $l(\alpha) = \sum_{i=1}^{m-1} n_i$ (see (2.5)). Consider the linear function

$$\operatorname{tr}_{p_{\mathbf{t}}}(x) = \frac{\operatorname{tr}(x)}{\operatorname{tr}(p_{\mathbf{t}})}, \qquad x \in p_{\mathbf{t}} \mathscr{H}_{r+f} p_{\mathbf{t}}, \tag{3.5}$$

where tr( ) is the Markov trace on $\mathscr{H} = \bigcup_{r=1}^{+\infty} \mathscr{H}_r$ with respect to the parameter $z = q^n(1-q)/(1-q^n)$. Note that $p_{\mathbf{t}}$ is the identity element in

$p_{\mathbf{t}}\mathscr{H}_{r+f}p_{\mathbf{t}}$. Since tr( ) is the Markov trace on $\mathscr{H}$, one can verify easily that $\mathrm{tr}_{p_{\mathbf{t}}}$ defined as in (3.5) is the trace function on $\bigcup_{i=1}^{+\infty} p_{\mathbf{t}}\mathscr{H}_{r+f}p_{\mathbf{t}}$, which satisfies (3.1(1–4)). Using (3.3)–(3.5), the weight with respect to $\tilde{\boldsymbol{\lambda}}$ is $s_{\tilde{\boldsymbol{\lambda}},n}/s_{\alpha,n}$.

From here onward, we always assume that

$$n = n_1 + n_2 + \cdots + n_m \qquad \text{with } n_i > r,\ 1 \le i \le m-1,\ n_m \in \mathbb{N}.$$

Since $\rho_{f,r}$ is an epimorphism, the trace $\mathrm{tr}_{p_{\mathbf{t}}}$ defined as in (3.5) results in a linear function tr( ) on $\tilde{\mathbf{H}}$,

$$\mathrm{tr}\,|_{\tilde{\mathbf{H}}_r}(x) = \mathrm{tr}_{p_{\mathbf{t}}}\big(\rho_{r,f}(x)\big), \qquad x \in \tilde{\mathbf{H}}_r. \tag{3.6}$$

(3.7) THEOREM. *Keep the notation above.*

(1) tr( ) *is the Markov trace on* $\tilde{\mathbf{H}}$ *with respect to the parameters* $z = q^n(1-q)/(1-q^n)$, *with* $n = \sum_{i=1}^m n_i$, $n_i > r$, $1 \le i \le m-1$, $n_m \in \mathbb{N}$, *and* $s_k = \mathrm{tr}(T_0^k)$, $0 \le k \le m-1$.

(2) *Let* α *be the partition defined as in* (2.5). *Then*

$$\mathrm{tr}\,|_{\tilde{\mathbf{H}}_r}(x) = \sum_{\boldsymbol{\lambda}\in\Lambda_m^+(r)} \frac{s_{\tilde{\boldsymbol{\lambda}},n}}{s_{\alpha,n}}\chi^{\boldsymbol{\lambda}}(x).$$

Proof. As mentioned below (3.5), the trace function $\mathrm{tr}_{p_{\mathbf{t}}}$ defined on $p_{\mathbf{t}}\mathscr{H}_{f+r}p_{\mathbf{t}}$ is the Markov trace with respect to the parameter $z = q^n(1-q)/(1-q^n)$. So, tr( ) defined on $\tilde{\mathbf{H}}$ satisfies the conditions (3.1(1–3)). By (2.8(2)),

$$\mathrm{tr}\,|_{\tilde{\mathbf{H}}_r}(x) = \sum_{\boldsymbol{\lambda}\in\Lambda_m^+(r)} \omega_{\boldsymbol{\lambda}}\,\chi^{\boldsymbol{\lambda}}(x) = \mathrm{tr}_{p_{\mathbf{t}}}\big(\rho_{r,f}(x)\big),$$

where $\chi^{\boldsymbol{\lambda}}$ is the irreducible character with respect to the simple module $p_{\mathbf{t}}V_{\tilde{\boldsymbol{\lambda}}}$. So, $\omega_{\boldsymbol{\lambda}} = s_{\tilde{\boldsymbol{\lambda}},n}/s_{\alpha,n}$. We claim that

$$\begin{aligned}\mathrm{tr}\Big(p_{\mathbf{t}}T_w p_{\mathbf{t}} T_{f+r+1,f+1}\big(T_{f+1,1}T_{1,f+1}\big)^k T_{f+r+1,f+1}^{-1}\Big)&\\ = \mathrm{tr}_{p_{\mathbf{t}}}(p_{\mathbf{t}}T_w p_{\mathbf{t}})\,\mathrm{tr}\Big(p_{\mathbf{t}}\big(T_{f+1,1}T_{1,f+1}\big)^k\Big),& \qquad w \in \mathfrak{S}_{f+r}.\end{aligned} \tag{3.8}$$

It is easy to see that (3.8) holds if $w = e$, where e is the identity element in $\mathfrak{S}_{r+f}$. Suppose $l(w) \ge 1$, where $l(w)$ is the length of w. We prove (3.8) by induction on r and $l(w)$.

Suppose $r = 1$. Since z_α is the minimal central idempotent with respect to the irreducible representation of $\mathscr{H}_f$ indexed by α, $z_\alpha\mathscr{H}_f z_\alpha \cong \mathrm{End}_{\mathscr{H}_f}(z_\alpha\mathscr{H}_f) \cong \mathbb{C}$ and $z_\alpha\mathscr{H}_f z_\alpha = \mathbb{C}z_\alpha$. By (1.5),

$$p_{\mathbf{t}}\mathscr{H}_f p_{\mathbf{t}} = \mathbb{C}p_{\mathbf{t}}. \tag{3.9}$$

If $l(w) = 1$, then $w = s_i$, $1 \le i \le f$. By [W1, (2.3e)] and (3.9), $p_{\mathbf{t}} T_w p_{\mathbf{t}} = c p_{\mathbf{t}}$ for some $c \in \mathbb{C}$, forcing (3.8) to hold. Suppose $l(w) > 1$. Then $w = s_{i,f+1} x$, where $x \in \mathfrak{S}_f$ with $l(w) = l(s_{i,f+1}) + l(x)$. The case $i = f+1$ follows from (3.9) since $T_w \in \mathscr{H}_f$. If $i < f+1$, then

$$\begin{aligned}
&\operatorname{tr}\Big(p_{\mathbf{t}} T_{i,f+1} T_x p_{\mathbf{t}} T_{f+1} (T_{f+1,1} T_{1,f+1})^k T_{f+1}^{-1} \Big) \\
&\quad = \operatorname{tr}\Big(p_{\mathbf{t}} T_{i,f} \underline{T_{f+1}^{-1} T_f T_{f+1}} T_x p_{\mathbf{t}} (T_{f+1,1} T_{1,f+1})^k \Big) \qquad \text{by } 3.1(1), 2.1(5) \\
&\quad = \operatorname{tr}\Big(p_{\mathbf{t}} T_{i,f} T_f \underline{T_{f+1}} T_f^{-1} T_x p_{\mathbf{t}} (T_{f+1,1} T_{1,f+1})^k \Big) \qquad \text{by } 2.1(4) \\
&\quad = z \operatorname{tr}\Big(p_{\mathbf{t}} T_{i,f} T_x p_{\mathbf{t}} (T_{f+1,1} T_{1,f+1})^k \Big) \qquad \text{by } 3.1(3), 3.1(1) \\
&\quad = z \operatorname{tr}_{p_{\mathbf{t}}}(p_{\mathbf{t}} T_{i,f} T_x p_{\mathbf{t}}) \operatorname{tr}\Big(p_{\mathbf{t}} (T_{f+1,1} T_{1,f+1})^k \Big) \qquad \text{by (3.9)} \\
&\quad = \operatorname{tr}_{p_{\mathbf{t}}}(p_{\mathbf{t}} T_w p_{\mathbf{t}}) \operatorname{tr}\Big(p_{\mathbf{t}} (T_{f+1,1} T_{1,f+1})^k \Big) \qquad \text{by } 3.1(3), w = s_{i,f+1} x.
\end{aligned}$$

This completes the proof of (3.8) for $r = 1$. Suppose $r > 1$ and $l(w) \ge 1$. Then $w = s_{i,f+r} x$, where $x \in \mathfrak{S}_{f+r-1}$ with $l(w) = l(s_{i,f+r}) + l(x)$. If $i = f + r$, then $T_w \in \mathscr{H}_{f+r-1}$ and

$$\begin{aligned}
&\operatorname{tr}\Big(p_{\mathbf{t}} T_w p_{\mathbf{t}} T_{f+r+1,f+1} (T_{f+1,1} T_{1,f+1})^k T_{f+r+1,f+1}^{-1} \Big) \\
&\quad = \operatorname{tr}\Big(p_{\mathbf{t}} T_w p_{\mathbf{t}} T_{f+r,f+1} (T_{f+1,1} T_{1,f+1})^k T_{f+r,f+1}^{-1} \Big) \qquad \text{by } 3.1(1), 2.1(5) \\
&\quad = \operatorname{tr}_{p_{\mathbf{t}}}(p_{\mathbf{t}} T_w p_{\mathbf{t}}) \operatorname{tr}\Big(p_{\mathbf{t}} (T_{f+1,1} T_{1,f+1})^k \Big) \qquad \text{by induction assumption.}
\end{aligned}$$

Suppose $i < r + f$. We have

$$\begin{aligned}
&\operatorname{tr}\Big(p_{\mathbf{t}} T_{i,f+r} T_x p_{\mathbf{t}} T_{r+f} T_{f+r,f+1} (T_{f+1,1} T_{1,f+1})^k T_{f+r,f+1}^{-1} T_{r+f}^{-1} \Big) \\
&\quad = \operatorname{tr}\Big(p_{\mathbf{t}} T_{i,f+r-1} \underline{T_{f+r}^{-1} T_{f+r-1} T_{f+r}} T_x p_{\mathbf{t}} T_{f+r,f+1} (T_{f+1,1} T_{1,f+1})^k T_{f+r,f+1}^{-1} \Big) \\
&\quad = \operatorname{tr}\Big(p_{\mathbf{t}} T_{i,f+r-1} T_{f+r-1} \underline{T_{f+r}} T_{f+r-1}^{-1} T_x p_{\mathbf{t}} T_{f+r,f+1} (T_{f+1,1} T_{1,f+1})^k T_{f+r,f+1}^{-1} \Big) \\
&\quad = z \operatorname{tr}\Big(p_{\mathbf{t}} T_{i,f+r-1} T_x p_{\mathbf{t}} T_{f+r,f+1} (T_{f+1,1} T_{1,f+1})^k T_{f+r,f+1}^{-1} \Big) \\
&\quad = z \operatorname{tr}_{p_{\mathbf{t}}}(p_{\mathbf{t}} T_{i,f+r-1} T_x p_{\mathbf{t}}) \operatorname{tr}\Big(p_{\mathbf{t}} (T_{f+1,1} T_{1,f+1})^k \Big) \\
&\qquad\qquad \text{by the induction assumption} \\
&\quad = \operatorname{tr}_{p_{\mathbf{t}}}(p_{\mathbf{t}} T_w p_{\mathbf{t}}) \operatorname{tr}\Big(p_{\mathbf{t}} (T_{f+1,1} T_{1,f+1})^k \Big) \qquad \text{by } 3.1(3).
\end{aligned}$$

This completes the proof of (3.8). For any $h \in \tilde{\mathbf{H}}_r$, $\rho_{f,r}(h) \in p_{\mathbf{t}}\mathscr{H}_{f+r}p_{\mathbf{t}}$. Since $\{T_w \mid w \in \mathfrak{S}_{f+r}\}$ forms a $\mathbb{C}$-base of $\mathscr{H}_{f+r}$, by (3.8), we have

$$\begin{aligned}\operatorname{tr}\Big(\rho_{r,f}(h)T_{f+r+1,f+1}\big(T_{f+1,1}T_{1,f+1}\big)^k T^{-1}_{f+r+1,f+1}\Big)\\ = \operatorname{tr}_{p_{\mathbf{t}}}\big(\rho_{r,f}(h)\big)\operatorname{tr}\Big(p_{\mathbf{t}}\big(T_{f+1,1}T_{1,f+1}\big)^k\Big).\end{aligned}$$

By (3.6), we have $\operatorname{tr}(ht_r'^k) = s_k \operatorname{tr}(h)$ for any $h \in \tilde{\mathbf{H}}_r$, proving 3.1(4). ∎

Suppose $u_i q^{k_{ij}} - u_j \neq 0$, for $1 \le i < j \le m$ and $1 - n_i \le k_{ij} \le n_j - 1$, and $n_i > r$, $1 \le i \le m-1$. For any $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in \Lambda_m^+(r)$, let $m_i = \sum_{j=1}^{l(\lambda^{(i)})}(j-1)\lambda_j^{(i)}$ and $n = \sum_{i=1}^m n_i$. Let

$$W_{\boldsymbol{\lambda}}(q, u_1, \dots, u_m) = q^{\sum_{i=1}^m m_i}\left(\frac{1-q}{1-q^n}\right)^r \prod_{i=1}^m \prod_{1 \le j < k \le n_i} \frac{1 - q^{\lambda_j^{(i)} - \lambda_k^{(i)} + k - j}}{1 - q^{k-j}}$$
$$\times \prod_{1 \le i < j \le m} \prod_{k=1}^{n_i} \prod_{l=1}^{n_j} \frac{u_i q^{\lambda_k^{(i)} - k} - u_j q^{\lambda_l^{(j)} - l}}{u_i q^{-k} - u_j q^{-l}}.$$

It is easy to see that $W_\lambda(q, u_1, \dots, u_m)$ is a rational function with $u_i q^{-k} - u_j q^{-l} \neq 0$ for all $1 \le i < j \le m$, $1 \le k \le n_i$, and $1 \le l \le n_j$.

(3.12) THEOREM. *Let $n_i \in \mathbb{N}$ with $n_i > r$, $1 \le i \le m-1$, and let $n = \sum_i n_i$. Define* $\operatorname{tr}(\ )$: $\mathbf{H} \to \mathbb{C}$ *to be the linear function with*

$$\operatorname{tr}|_{\mathbf{H}_r}(x) = \sum_{\boldsymbol{\lambda} \in \Lambda_m^+(r)} W_\lambda \chi^{\boldsymbol{\lambda}}(x) \qquad \text{for any } x \in \mathbf{H}_r.$$

Then $\operatorname{tr}(\ )$ *is the Markov trace on* $\mathbf{H}$ *with respect to the parameters $z = q^n(1-q)/(1-q^n)$, and $s_k = \operatorname{tr}(T_0^k)$, $0 \le k \le m-1$, where*

$$s_k = \sum_{i=1}^n u_i^k \frac{1-q^{n_i}}{1-q^n} \prod_{j \neq i, 1 \le j \le m} \frac{u_j - u_i q^{n_j}}{u_j - u_i}. \tag{3.13}$$

Proof. Since $\chi^{\boldsymbol{\lambda}}$ is a trace function on $\mathbf{H}_r$, it is easy to see that $\operatorname{tr}|_{\mathbf{H}_r}$ satisfies 3.1(1). However, when $u_i = \tilde{u}_i$, $i = 1, \dots, m$ (see (2.4)), tr turns out to be the Markov trace function on $\tilde{\mathbf{H}}$ with respect to the parameters z, $s_k = \operatorname{tr}(T_0^k)$. A direct computation shows that $W_{\boldsymbol{\lambda}}(q, \tilde{u}_1, \dots, \tilde{u}_m) = \omega_{\boldsymbol{\lambda}}$. Now we prove that (3.1(2–4)) holds.

Let $F(q, u_1, \dots, u_{m-1})$ be a polynomial in indeterminate $q, u_1, \dots, u_m$. We claim that $F(q, u_1, \dots, u_{m-1}) \equiv 0$ if $F(q, \tilde{u}_1, \dots, \tilde{u}_{m-1}) = 0$ for all $\tilde{u}_i$, $1 \le i \le m-1$. In fact, we write

$$F(q, u_1, \dots, u_{m-1}) = \sum_i f_i(q, u_1, \dots, u_{m-2}) u_{m-1}^i,$$

where $f_i(q, u_1, \ldots, u_{m-2})$ is a polynomial in indeterminates $q, u_1, \ldots, u_{m-2}$. We hope to prove that $f_i(q, u_1, \ldots, u_{m-2}) \equiv 0$. Fix $u_i = \tilde{u}_i$, $1 \le i \le m-2$. Then $F(q, \tilde{u}_1, \ldots, \tilde{u}_{m-1}) = \sum_i f_i(q, \tilde{u}_1, \ldots, \tilde{u}_{m-2})\tilde{u}_{m-1}^i = 0$. By the identity theorem, $F(q, \tilde{u}_1, \ldots, \tilde{u}_{m-2}, u_{m-1}) \equiv 0$. So, $f_i(q, \tilde{u}_1, \ldots, \tilde{u}_{m-2}) = 0$. By induction, we have $f_i(q, u_1, \ldots, u_{m-2}) = 0$. Now, consider the function $\operatorname{tr}(1)(q, u_1, \ldots, u_{m-1}) - 1$. Since $W_{\boldsymbol{\lambda}}$ and $\chi^{\boldsymbol{\lambda}}$ are analytic rational functions, $\operatorname{tr}(x) = F_x(q, u_1, \ldots, u_m)/G_x(q, u_1, \ldots, u_m)$ for some polynomials $F_x(q, u_1, \ldots, u_m)$ and $G_x(q, u_1, \ldots, u_m)$ depending on x such that $G_x(q, u_1, \ldots, u_m) \neq 0$. Since $\operatorname{tr}(x)(q, \tilde{u}_1, \ldots, \tilde{u}_m)$ is the Markov trace on $\tilde{\mathbf{H}}$, $\operatorname{tr}(1)(q, \tilde{u}_1, \ldots, \tilde{u}_m) = 1$ and $F_1(q, \tilde{u}_1, \ldots, \tilde{u}_m) = G_1(q, u_1, \ldots, \tilde{u}_m)$. So,

$$F_1(q, u_1, \ldots, u_m) = G_1(q, u_1, \ldots, u_m)$$

and $\operatorname{tr}(1) = 1$. One can prove (3.1(3–4)) similarly.

Let $\gamma^{(i)} = ((0), \ldots, (0), (1), (0), \ldots, (0))$. Via (3.11),

$$W_{\gamma^{(i)}} = \frac{1 - q^{n_i}}{1 - q^n} \prod_{j \neq i, 1 \le j \le m} \frac{u_j - u_i q^{n_j}}{u_j - u_i}.$$

Now, (3.13) follows since $\chi^{\gamma^{(i)}}(T_0^k) = u_i^k$ (see (2.9)). ∎

ACKNOWLEDGMENT

It is my pleasure to thank Professor Lambropoulou for sending her reprint [L] to me, which motivated me to write this paper.

REFERENCES

[A1] S. Ariki, On the semi-simplicity of the Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$, *J. Algebra* **169** (1994), 216–225.

[A2] S. Ariki, On the classification of simple modules for cyclotomic Hecke algebras of type $G(m, 1, n)$ and Kleshchev multipartitions, preprint.

[AK] S. Ariki and K. Koike, A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and the construction of its irreducible representations, *Adv. Math.* **106** (1994), 216–243.

[BM] M. Broué and G. Malle, Zyklotomische Heckealgebren, *Astérisque* **212** (1993), 119–202.

[DJ] R. Dipper and G. James, Representations of Hecke algebras of general linear groups, *Proc. London Math. Soc.* (3) **52** (1986), 20–52.

[DJM] R. Dipper, G. James, and A. Mathas, Cyclotomic q-Schur algebras, *Math. Z.* **229** (1998), 385–416.

[DR1] J. Du and H. Rui, Ariki–Koike algebras with semi-simple bottoms, *Math. Z.* **234** (2000), 807–830.

[DR2] J. Du and H. Rui, Specht modules for Ariki-Koike algebras, *Comm. Algebra*, to appear.

[GIM] M. Geck, L. Iancu, and G. Malle, Weights of Markov traces and generic degrees, *Indag. Math.*, to appear.

[GL] M. Geck and S. Lambropoulou, Markov traces and Knot invariants related to Iwahori–Hecke algebras of type B, *J. Reine Angew. Math.* **482** (1997), 191–213.

[I] L. Iancu, Markov traces and generic degrees in type B_n, *J. Algebra*, to appear.

[J] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. Math.* **126** (1987), 335–388.

[KL] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

[L] S. Lambropoulou, Knot theory related to generalized and cyclotomic Hecke algebras of type B, *J. Knot Theory and Its Ramifications* **8** (1999), 621–658.

[M] G. Murphy, The representations of Hecke algebras of type A_n, *J. Algebra* **173** (1995), 97–121.

[Ma] A. Mathas, "Iwahori–Hecke Algebras and Schur Algebras of the Symmetric Group," University Lecture Series, Vol. 15, American Mathematical Society, Providence, RI, 1999.

[Mc] I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Oxford Mathematical Monographs, Clarendon Press/Oxford Univ. Press, New York, 1979.

[O] R. C. Orellana, Weights of Markov traces on Hecke algebras, *J. Reine. Angew. Math.* **508** (1999), 157–178.

[W1] H. Wenzl, Hecke algebras of type A_n and subfactors, *Invent. Math.* **92** (1988), 349–383.

[W2] H. Wenzl, Braids and invariants of 3 manifolds, *Invent. Math.* **114** (1993), 235–275.